Multiple Scattering Theory: 
Introduction and Practical tools

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Context

- Cours détaillé introductif, *B. Djafari-Rouhani*
- Métamatériaux dans l’industrie de l’acoustique audible : Cas du métaporeux, *C. Lagarrigue*
- Métamatériaux acoustiques, *J. Sánchez-Dehesa*
- Relation de dispersion - PWE, EPWE, *J. Vasseur*
- Métamatériaux et aspects perceptifs, *N. Côté*
- Technique d’homogénéisation, *A. Maurel*

**Sonic Crystals**

- Particular case of phononic crystal with a fluid as host medium.
- Made of rigid, penetrable or resonance scatterers.


Abramowitz & Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, 1964


...
Part I. Introduction

- What is scattering?
- One dimensional scattering

Part II. Scattering by circular rigid cylinders

- General background
- Scattering by a single circular cylinder
- Scattering by $N$ circular cylinders

Part III. Scattering by a periodic arrangement of circular cylinders

- Scattering of a plane incident by an array of rigid cylinders
- Reflection and transmission coefficients by an array of rigid cylinders
- Reflection and Transmission coefficients by a stack of gratings
- Band diagram calculation
Part I. Introduction

- What is scattering?
- One dimensional scattering
What is scattering?
Scattering is a general physical process where some forms of radiation, such as light, sound, or moving particles, are forced to deviate from a straight trajectory by one or more paths due to localized non-uniformities in the medium through which they pass. In conventional use, this also includes deviation of reflected radiation from the angle predicted by the law of reflection. Reflections that undergo scattering are often called diffuse reflections and unscattered reflections are called specular (mirror-like) reflections.

Is it more related to energy?
What is Single and Multiple Scattering?

When radiation is only scattered by one localized scattering center, this is called **single scattering**. It is very common that scattering centers are grouped together; in such cases, radiation may scatter many times, in what is known as **multiple scattering**. The **main difference** between the effects of single and multiple scattering is that single scattering can usually be treated as a random phenomenon, whereas **multiple scattering**, somewhat counterintuitively, can be modeled as a more **deterministic** process because the combined results of a large number of scattering events tend to average out. Multiple scattering can thus often be modeled well with **diffusion theory**.
Physical interpretation of the bandgap: Bragg interferences

\[ 2d \sin \theta = n\lambda. \]

In particular, only specularly reflected and transmitted waves are propagative in the surrounding medium for finite depth sonic crystals within the first Bragg bandgap.
One dimensional scattering
One dimensional scattering

Pressure field is splitted into upward and downward going waves:

\[
\begin{align*}
\begin{aligned}
 p_u^+ &= Rp_u^- + Tp_d^+ , \\
p_d^- &= Rp_d^+ + Tp_u^- ,
\end{aligned}
\end{align*}
\]

in case of reciproque and symetric scattering.

Outgoing waves
\[
\begin{bmatrix}
 p_u^+ \\
p_d^-
\end{bmatrix}
= 
\begin{bmatrix}
 R & T \\
 T & R
\end{bmatrix}
\begin{bmatrix}
 p_u^- \\
p_d^+
\end{bmatrix}
\]

Ingoing waves
\[
\begin{bmatrix}
 p_u^- \\
p_d^+
\end{bmatrix}
= 
\begin{bmatrix}
 R & T \\
 T & R
\end{bmatrix}^{-1}
\begin{bmatrix}
 p_u^+ \\
p_d^-
\end{bmatrix}
\]

\( SC \) eigenvalues are \( \lambda = (R \pm T) \): symetric and antisymetric problem.

- \(|\lambda_S|^2 \) \((|\lambda_A|^2)\) reflected energy in the (anti)symetric problem
- \( \alpha_S = 1 - |\lambda_S|^2 \) \((\alpha_A = 1 - |\lambda_A|^2)\) absorbed energy in the (anti)symetric problem
- \(|R|^2 = \left\| \frac{\lambda_S + \lambda_A}{2} \right\|^2 \) and \(|T|^2 = \left\| \frac{-\lambda_S + \lambda_A}{2} \right\|^2 \) reflected and transmitted energy by the global system
- \( \alpha = \frac{\alpha_S + \alpha_A}{2} \) absorbed energy by the global system
Part II. Scattering by circular rigid cylinders

- General background
- Scattering by a single circular cylinder
- Scattering by $N$ circular cylinders
General background
Helmholtz equation in cylindrical coordinates

Helmholtz equation \((e^{-i\omega t} \text{ time convention})\)

\[
(\Delta + k^2) p(r) = 0, \forall r \in \mathbb{R}^2.
\]

In cylindrical coordinate system, \(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\) and the Helmholtz equation reads as

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k^2 \right) p(r) = 0, \forall r \in \mathbb{R}^2.
\]

Separation of variables: \(p(r) = F(\theta)G(r)\), with \(F(\theta) = F(\theta + 2\pi n)\), \(\forall n \in \mathbb{Z}\) (\(\theta\) periodic) + geometry

\[
\frac{1}{F(\theta)} \frac{\partial F(\theta)^2}{\partial \theta^2} = -k^2 r^2 - \left( \frac{r^2}{G(r)} \frac{\partial^2 G(r)}{\partial r} + \frac{r}{G(r)} \frac{\partial G(r)}{\partial r} \right), \forall r \in \mathbb{R}^2.
\]
Solution of the Helholtz equation

\[
\begin{align*}
\frac{1}{F(\theta)} \frac{\partial^2 F(\theta)}{\partial \theta^2} &= -\nu^2, \\
F(\theta) &= F(\theta + 2n\pi), \quad \forall n \in \mathbb{Z}, \quad \Rightarrow F(\theta) = \sum_{n \in \mathbb{Z}} A e^{in\theta} + B e^{-in\theta}. \\
\end{align*}
\]

For fixed \( n \), introducing \( \alpha = kr \), \( G_n(\alpha) \) satisfies the Bessel’s equation

\[
\frac{\partial^2 G_n(\alpha)}{\partial \alpha^2} + \frac{1}{\alpha} \frac{\partial G_n(\alpha)}{\partial \alpha} + \left( 1 - \frac{n^2}{\alpha^2} \right) G_n(\alpha) = 0,
\]

whose solution is

\[
G_n(\alpha) = C \underbrace{J_n(\alpha)}_{\text{Bessel function of 1st kind}} + D \underbrace{H_n^{(1)}(\alpha)}_{\text{Hankel function of 1st kind}}.
\]

where \( p(r) = \sum_{n \in \mathbb{Z}} \left( A_n J_n(kr) + B_n H_n^{(1)}(kr) \right) e^{in\theta} \),

because \( J_{-n}(kr) = (-1)^n J_n(kr) \) and \( H_{-n}^{(1)}(kr) = (-1)^n H_n^{(1)}(kr) \).
Physical meaning of the solution

\[ p(r) = \sum_{n \in \mathbb{Z}} A_n J_n(kr)e^{in\theta} \]
\[ p(r) = \sum_{n \in \mathbb{Z}} B_n H_n^{(1)}(kr)e^{in\theta} \]

Remark: the solution could alternatively be sought in the form
\[ p(r) = \sum_{n \in \mathbb{Z}} \left( A'_n J_n(kr) + B'_n Y_n(kr) \right) e^{in\theta} . \]

Bessel function of 2nd kind

The scattering problem also implies to relate \( B_n \) to \( A_n \), \( \forall n \in \mathbb{Z} \):
\[ B = SCA. \]
Scattering by a single cylinder
Scattering of a plane incident wave by a rigid cylinder

Look for $p^0(r)$, $\forall r \in \Omega^0$,

\[
\begin{cases}
(\nabla + k^2)p^0(r) = 0, \\
\frac{\partial p^0(r)}{\partial r} \bigg|_{r=R} = 0.
\end{cases}
\]

wherein $p^i(r) = e^{ik^i_1x_1 - ik^i_2x_2}$, with $k^i_1 = -k \cos(\theta^i)$ and $k^i_2 = \sqrt{k^2 - (k^i_1)^2}$, with $\text{Re}(k^i_2) \geq 0$.

Boundary conditions:

$V^0_r(R) = 0 \Rightarrow \left. \frac{\partial p^0}{\partial r} \right|_{r=R} = 0.$

The pressure field in $\Omega^0$ takes the following form

\[
p(r) = \sum_{m \in \mathbb{Z}} A^m J_m(kr)e^{im\theta} + \sum_{n \in \mathbb{Z}} B^n H_n^{(1)}(kr)e^{in\theta}.
\]

Incident field \hspace{2cm} Scattered field

Remark: $\int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi \delta_{nm}$, i.e., $e^{in\theta}$ is an orthogonal basis.
Expression of the incident field in $\mathcal{C}$

$$k^i = \left[ \begin{array}{c} k_1^i = -k \cos(\theta^i) \\ k_2^i = k \sin(\theta^i) \end{array} \right], \quad x = \left[ \begin{array}{c} x_1 = r \cos(\theta) \\ x_2 = r \sin(\theta) \end{array} \right].$$

$$p^i(r) = e^{ik_1^i x_1 - ik_2^i x_2}$$
$$= e^{-ik \cos(\theta^i) r \cos(\theta) - ik \sin(\theta^i) r \sin(\theta)}$$
$$= e^{-ikr \left[ \cos(\theta^i) \cos(\theta) + \sin(\theta^i) \sin(\theta) \right]}$$
$$= e^{-ikr \cos(\theta - \theta^i)}.$$

Refering to Abramowitz & Stegun, 1964:

$$e^{-ikr \cos(\theta - \theta^i)} = \sum_{m \in \mathbb{Z}} (-i)^m J_m(kr) e^{im(\theta - \theta^i)},$$

so the incident field may be written as

$$p^i(r) = \sum_{m \in \mathbb{Z}} \left\{ (-i)^m e^{-im\theta^i} \overline{A}_m \right\} J_m(kr) e^{in\theta}.$$
Application of the BC and solution of the problem

\[ p(r) = \sum_{m \in \mathbb{Z}} A_m J_m(kr)e^{im\theta} + \sum_{n \in \mathbb{Z}} B_n H_n^{(1)}(kr)e^{in\theta}. \]

The normal derivative with respect of \( r \) reads as

\[ \frac{\partial p(r)}{\partial r} = \sum_{m \in \mathbb{Z}} kA_m J_m(kr)e^{im\theta} + \sum_{n \in \mathbb{Z}} kB_n H_n^{(1)}(kr)e^{in\theta}, \]

where \( \chi_n(x) = \frac{\partial \chi_n(x)}{\partial x} = \frac{\chi_{n-1}(x) - \chi_{n+1}(x)}{2} \).

Introducing \( \alpha = kr \) and making use of \( \int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi \delta_{nm} \) after projection \( R \times \int_0^{2\pi} \frac{\partial p(r)}{\partial r} \bigg|_{r=R} e^{-il\theta} d\theta = 0 \), we get:

\[ B_n = -\frac{\dot{J}_n(\alpha)}{\dot{H}_n^{(1)}(\alpha)} A_n = \underset{\text{Scattering coefficient}}{\text{SC}_n} A_n, \]

and finally \( p(r) = \sum_{n \in \mathbb{Z}} A_n \left( J_n(kr) + SC_n H_n^{(1)}(kr) \right) e^{in\theta}. \)
Penetrable scatterers

- when the cylindrical scatterer is penetrable (fluid), the pressure field in $\Omega^{[1]}$ reads as (Rayleigh hypothesis):

$$ p^{[1]}(r) = \sum_{m \in \mathbb{Z}} C_m J_n(kr)e^{in\theta}e^{in\theta}. $$

Application of the BC (after projection $\int_0^{2\pi} \cdot e^{-il\theta} d\theta$) leads to

$$ B_n = \frac{\beta^{[1]} J_n(\alpha^{[1]} J_n(\alpha^{[0]}) - \beta^{[0]} J_n(\alpha^{[0]} J_n(\alpha^{[1]}))}{\beta^{[0]} H_n^{(1)}(\alpha^{[0]} J_n(\alpha^{[1]})) - \beta^{[1]} H_n^{(1)}(\alpha^{[0]} J_n(\alpha^{[1]}))} A_n = SC_n A_n, $$

where $\alpha^{[j]} = k^{[j]} R$, $\beta^{[j]} = \alpha^{[j]} / \rho^{[j]}$, $j = 0, 1$.

Remark: the low frequency approximation reads as $(O(\alpha^{[0]})^2)$:

$$ SC_0 \approx \frac{i\pi}{4} (\alpha^{[0]})^2 \left( 1 - \frac{K^{[0]}}{K^{[1]}} \right), \quad SC_{\pm 1} \approx \frac{i\pi}{4} (\alpha^{[0]})^2 \frac{\rho^{[1]} - \rho^{[0]}}{\rho^{[0]} + \rho^{[1]}}. $$

Scattering of a cylindrical incident wave by a rigid cylinder

Look for $p^{[0]}(r)$, $\forall r \in \Omega^{[0]}$,

$$\begin{cases} (\nabla + k^2)p^{[0]}(r) = 0, \\ p^{[0]}(r) - p^i(r) \sim \text{outgoing waves}, \end{cases}$$

wherein $p^i(r_s) = \frac{i}{4} H_0^{(1)}(kr_s)$.

Boundary conditions: $\left. \frac{\partial p^{[0]}}{\partial r} \right|_{r=R} = 0$.

Notation:
- superscript indicates the object
- subscript indicates the object the coordinate system is attached to

The pressure field in $\Omega^{[0]}$ takes the following form

$$p(r) = \frac{i}{4} H_0^{(1)}(kr_s) + \sum_{n \in \mathbb{Z}} B_n H_n^{(1)}(kr)e^{in\theta}.$$
Graf’s addition theorem

\[ H_n^{(1)}(kr_2)e^{in\theta_2} = \begin{cases} \sum_{q \in \mathbb{Z}} e^{i(n-q)\theta_1^2}H_{q-n}^{(1)}(kr_1^2)J_q(kr_1)e^{iq\theta_1}, & \text{if } r_1 < r_1^2 \\ \sum_{q \in \mathbb{Z}} e^{i(n-q)\theta_1^2}J_{q-n}(kr_1^2)H_q^{(1)}(kr_1)e^{iq\theta_2}, & \text{if } r_1 > r_1^2 \end{cases} \]

Remark: may also be found with \( \theta_1^{2'} = \theta_1^2 + \pi \).
Solution of the scattering problem

Applying the Graf’s theorem, we end with

\[ p^i(r) = \begin{cases} 
\sum_{n \in \mathbb{Z}} \frac{i}{4} e^{-in\theta^s} J_n(kr^s) H_n^{(1)}(kr) e^{in\theta}, & \text{for } r > r^s, \\
\sum_{n \in \mathbb{Z}} \frac{i}{4} e^{-in\theta^s} H_n^{(1)}(kr^s) J_n(kr) e^{in\theta}, & \text{for } r < r^s,
\end{cases} \]

so the problem reads as

\[ p(r) = \sum_{n \in \mathbb{Z}} \left( A_n J_n(kr) + B_n H_n^{(1)}(kr) \right) e^{in\theta}, \text{ for } r < r^s. \]

Or, we show previously that

\[ B_n = -\frac{\dot{J}_n(\alpha)}{\dot{H}_n^{(1)}(\alpha)} A_n = SC_n A_n. \]

\[ p(r) = \frac{i}{4} H_0^{(1)}(kr_s) + \sum_{n \in \mathbb{Z}} SC_n A_n H_n^{(1)}(kr) e^{in\theta}. \]
Scattering by a $N$ cylinders
Scattering of a plane incident wave by 2 rigid cylinders

Boundary conditions:
\[ \frac{\partial p[0]}{\partial r} \bigg|_{r_1 = R^{(1)}} = 0, \]
\[ \frac{\partial p[0]}{\partial r} \bigg|_{r_2 = R^{(2)}} = 0. \]

Look for \( p[0](r) \), \( \forall r \in \Omega^{[0]} \),

\[
\begin{cases}
(\nabla + k^2)p[0](r) = 0, \\
p[0](r) - p^i(r) \sim \text{outgoing waves,}
\end{cases}
\]

wherein \( p^i(r) = e^{ik^i_1x_1 - ik^i_2x_2} \), with \( k^i_1 = -k \cos(\theta^i) \) and
\( k^i_2 = \sqrt{k^2 - (k^i_1)^2} \), with \( \text{Re}(k^i_2) \geq 0 \).
Scattering of a plane incident wave by 2 rigid cylinders

Boundary conditions:
\[ \frac{\partial p[0]}{\partial r} \bigg|_{r_1=R(1)} = 0, \]
\[ \frac{\partial p[0]}{\partial r} \bigg|_{r_2=R(2)} = 0. \]

The pressure field in \( \Omega[0] \) takes the following form

\[
p(r) = \sum_{m \in \mathbb{Z}} (-i)^m J_m(kr) e^{im(\theta-\theta^i)} + \sum_{n \in \mathbb{Z}} B_n^{(1)} H_n^{(1)}(kr_1) e^{in\theta_1} + \sum_{q \in \mathbb{Z}} B_q^{(2)} H_q^{(1)}(kr_2) e^{iq\theta_2}.
\]

Global coordinate system

Coordinate system \( C_1 \)

Coordinate system \( C_2 \)
Expression of the field in $C_1$

\[
p(r) = e^{ik_1 x_1 - ik_2 x_2} + \sum_{n \in \mathbb{Z}} B_n^{(1)} H_n^{(1)}(kr_2) e^{in\theta_2} + \sum_{q \in \mathbb{Z}} B_q^{(2)} H_1^{(1)}(kr_2) e^{iq\theta_2}.
\]

- Incident field

\[
p^i(r) = e^{ik^i \cdot r} = e^{ik^i \cdot (r + r_1)} = e^{ik^i r_1} \times e^{ik^i \cdot r_1} = e^{-ik^i r_1 \cos(\theta^1 - \theta^i)} \times \sum_{n \in \mathbb{Z}} \left(-i\right)^n J_n(kr_1) e^{in(\theta_1 - \theta^i)} = \sum_{n \in \mathbb{Z}} A_n^{1i} J_n(kr_1) e^{in\theta_1}.
\]
Expression of the field in $C_1$

\[
p(r) = e^{ik_1^i x_1 - ik_2^i x_2} + \sum_{n \in \mathbb{Z}} B_n^{(1)} H_n^{(1)}(kr_2) e^{in\theta_2} + \sum_{q \in \mathbb{Z}} B_q^{(2)} H_q^{(1)}(kr_2) e^{iq\theta_2}.
\]

- Scattered field by the cylinder 2
  Graf's theorem, for $r_1 < r_2^2 - R^{(2)}$:

\[
H_q^{(1)}(kr_2) e^{iq\theta_2} = \sum_{n \in \mathbb{Z}} e^{i(q-n)\theta_1^2} H_{n-q}^{(1)}(kr_1^2) J_n(kr_1) e^{in\theta_1},
\]

so we got,

\[
p_{\text{scat}}^{(2)}(r_1) = \sum_{n \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} B_q^{(2)} e^{i(q-n)\theta_1^2} H_{n-q}^{(1)}(kr_1^2) J_n(kr_1) e^{in\theta_1}, \text{ for } r_1 < r_2^2 - R^{(2)}.
\]
Expression of the field in $C_1$, for $r_1 < r_1^2 - R(2)$

\[
p(r_1) = \sum_{n \in \mathbb{Z}} A_n^{1i} J_n(kr_1) e^{in\theta_1} \qquad \text{Incident field}
\]

\[
+ \sum_{n \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} B_q^{(2)} e^{i(q-n)\theta_1^2} H_n^{(1)}(kr_1^2) J_n(kr_1) e^{in\theta_1} \qquad \text{Scattered field by 2}
\]

\[
+ \sum_{n \in \mathbb{Z}} B_n^{(1)} H_n^{(1)}(kr_1) e^{in\theta_1} \qquad \text{Scattered field by 1}
\]

Keeping in mind that \( \int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi \delta_{nm} \), this field may be written as

\[
p(r_1) = \sum_{n \in \mathbb{Z}} \left( A_n^{1i} + \sum_{q \in \mathbb{Z}} B_q^{(2)} e^{i(q-n)\theta_1^2} H_n^{(1)}(kr_1^2) \right) J_n(kr_1) + B_n^{(1)} H_n^{(1)}(kr_1) e^{in\theta_1}.
\]
Solution of the problem

Once again, we have

\[ B_n^{(1)} = -\frac{\dot{J}_n(\alpha^1)}{\dot{H}_n^{(1)}(\alpha^1)} A_n^1 = SC_n^1 A_n^1 \]

\[ = SC_n^1 \left( A_n^{1i} + \sum_{q \in \mathbb{Z}} B_q^{(2)} e^{i(q-n)\theta_1^2} H_n^{(1)}(kr_1^2) \right), \]

which may be written in matrix form

\[ B_1^1 = A_1^1 + C_2^1 B_2^2. \]

Similarly, we can express the field in \( C_2 \) for \( r_2 < r_1^2 - R^{(1)} \) and we get:

\[ B_2^2 = A_2^2 + C_2^1 B_1^1. \]

Finally, the final system reads as:

\[
\begin{bmatrix}
\text{Id} & -C_2^1 \\
-C_2^1 & \text{Id}
\end{bmatrix}
\begin{bmatrix}
B_1^1 \\
B_2^2
\end{bmatrix}
= \begin{bmatrix}
A_1^1 \\
A_2^2
\end{bmatrix}.
\]
Warning! Take care with field representation domains

For the solution of the problem we expressed the fields in both
\( r_1 < r_1^2 - R^{(2)} \) and \( r_2 < r_2^2 - R^{(1)} \), but we should keep in mind that

\[
p(r) = \sum_{n \in \mathbb{Z}} (-i)^n J_n(kr) e^{in(\theta - \theta')} + B_n^{(1)} H_n^{(1)}(kr_1) e^{in\theta_1} + B_n^{(2)} H_n^{(1)}(kr_2) e^{in\theta_2}, \forall r \in \Omega^{[0]}.
\]
Scattering of a plane incident wave by $N$ cylinders

Boundary conditions:

$$\left. \frac{\partial p^{[0]}}{\partial r_j} \right|_{r_j=R^{(j)}} = 0, \; j \in J$$

Look for $p^{[0]}(r), \; \forall r \in \Omega^{[0]}$, 

$$\begin{cases} 
(\nabla + k^2)p^{[0]}(r) = 0, \\
+ p^{[0]}(r) - p^{i}(r) \sim \text{outgoing waves},
\end{cases}$$

wherein $p^{i}(r) = e^{ik_{1}^{i}x_1 - ik_{2}^{i}x_2}$, with $k_{1}^{i} = -k \cos (\theta^{i})$ and 

$k_{2}^{i} = \sqrt{k^2 - (k_{1}^{i})^2}$, with $\text{Re}(k_{2}^{i}) \geq 0$. 

J.-P. Groby

Multiple Scattering Theory
Scattering of a plane incident wave by $N$ cylinders

Boundary conditions:
\[
\left. \frac{\partial p^0}{\partial r_j} \right|_{r_j = R(j)} = 0, \quad j \in J
\]

The pressure field in $\Omega^0$ takes the following form

\[
P(r) = \sum_{n \in \mathbb{Z}} (-i)^n J_n (kr) e^{in(\theta - \theta^i)} + \sum_{j \in J} \sum_{n \in \mathbb{Z}} B_n^{(j)} H_n^{(1)} (kr_j) e^{in\theta_j}.
\]

Global coordinate system
Solution of the problem

- Express the field around each j-th cylinder $\forall r_j < \min_{o \neq j} (r^o_j - R^o(j))$:

$$p(r_j) = \sum_{n \in \mathbb{Z}} \left( \mathcal{A}^{(j)}_n + \sum_{o \in \mathcal{J} \neq j} \sum_{q \in \mathbb{Z}} B_{q}^{(o)} e^{i(q-n)\theta^o_j} H_{n-q}^{(1)}(kr^o_j) \right) J_n(kr_j) + B_{n}^{(1)} H_{n}^{(1)}(kr_j) e^{in\theta_j}.$$

- Apply the BC on the j-th cylinder:

$$B^{(j)}_n = SC^j_n \left( \mathcal{A}^{(j)}_n + \sum_{o \in \mathcal{J} \neq j} \sum_{q \in \mathbb{Z}} B_{q}^{(o)} e^{i(q-n)\theta^o_j} H_{n-q}^{(1)}(kr^o_j) \right).$$

- Final system for the solution of $B^{(j)}_n$, $\forall n \in \mathbb{Z}$ and $\forall j \in \mathcal{J}$

$$\begin{bmatrix}
\text{Id} & -C^2_1 & \cdots & -C^{J-1}_{1} & -C^J_1 \\
-C^2_1 & \text{Id} & \cdots & -C^{J-1}_{2} & -C^J_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-C^1_{J-1} & -C^{2}_{J-1} & \cdots & \text{Id} & -C^{J}_{J-1} \\
-C^1_{J} & -C^{2}_{J} & \cdots & -C^{J-1}_{J} & \text{Id}
\end{bmatrix}
\begin{bmatrix}
B^1 \\
B^2 \\
\vdots \\
B^{J-1} \\
B^J
\end{bmatrix}
= \begin{bmatrix}
A^1 \\
A^2 \\
\vdots \\
A^{J-1} \\
A^J
\end{bmatrix}.$$
Replacing the expression of $A_n^{ji}$ by $A_n^{ji} \equiv \frac{i}{4} e^{-in\theta^s_j} H_n^{(1)}(kr^s_j)$, enable the calculation of $B^j$, $j \in \mathcal{J}$ when the configuration is excited by a line source. In other words, you calculate the Green’s function of the system!

⇒ usefull to calculate the density of state Asatryan et al., Waves Random Media, 2003

Sum are truncated in practice and reads

$$\sum_{m=-M}^{M}, \text{ with}$$

$$M = \text{int} \left(4.05(kR)^{1/3} + kR\right) + \text{security coefficient},$$

Barber and Hill, 1990
Example: 77 element finite dimension sonic crystal

We should use the pointing vector instead of the pressure field.
Part III. Scattering by a periodic arrangement of circular cylinders

- Scattering of a plane incident by an array of rigid cylinders
- Reflection and transmission coefficients by an array of rigid cylinders
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- Band diagram calculation
Scattering of a plane incident by an array of rigid cylinders
Scattering by an array of rigid cylinders

Look for $p^0(r)$, $\forall r \in \Omega^0$,

$$\left\{ \begin{array}{l}
(\nabla + k^2)p^0(r) = 0, \\
+ p^0(r) - p^i(r) \sim \text{outgoing waves},
\end{array} \right.$$ 

wherein $p^i(r) = e^{ik^i_1x_1 - ik^i_2x_2}$, with $k^i_1 = -k \cos(\theta^i)$ and $k^i_2 = \sqrt{k^2 - (k^i_1)^2}$, with $\text{Re}(k^i_2) \geq 0$. 
Scattering by an array of rigid cylinders

Boundary conditions:
\[ \frac{\partial p[0]}{\partial r} \bigg|_{r=R} = 0. \]

The field is quasi-periodic (Floquet-Bloch condition):
\[ p[0](x_1 + nd, x_2) = p[0](x_1, x_2)e^{i k_i nd}, \ \forall x \in \mathbb{R}^2 \text{ and } \forall n \in \mathbb{Z}. \]

⇒ It is sufficient to determine the field in the unit cell \( \mathcal{C} \).
The pressure field in $\Omega^0$ takes the following form

$$p^{[0]}(r) = \sum_{n \in \mathbb{Z}} (-i)^n J_n(kr) e^{in(\theta - \theta^i)} + \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B_n^{(j)} H_n^{(1)}(kr_j) e^{in\theta_j}.$$ 

The periodicity implies

$$B_n^{(j)} = B_n^{(0)} e^{ijk_1^i d}.$$
The scattered field may be written in the form

\[ p_{\text{scat}}^{[0]}(r) = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B_n^{(0)} e^{i j k_1^i d} H_n^{(1)}(k r_j) e^{i n \theta_j} \]

\[ = \sum_{n \in \mathbb{Z}} B_n^{(0)} H_n^{(1)}(k r_0) e^{i n \theta_0} \]

\[ + \sum_{j < 0} \sum_{n \in \mathbb{Z}} B_n^{(0)} e^{i j k_1^i d} H_n^{(1)}(k r_j) e^{i n \theta_j} + \sum_{j > 0} \sum_{n \in \mathbb{Z}} B_n^{(0)} e^{i j k_1^i d} H_n^{(1)}(k r_j) e^{i n \theta_j}. \]

Applying the Graf’s theorem \((r_0 < d - R)\) leads to

\[ H_n^{(1)}(k r_j) e^{i n \theta_j} = \begin{cases} \sum_{q \in \mathbb{Z}} (-1)^{n-q} H_{q-n}^{(1)}(k |j| d) J_q(k r_0) e^{i q \theta_0}, & \text{for } j < 0, \\ \sum_{q \in \mathbb{Z}} H_{q-n}^{(1)}(k j d) J_q(k r_0) e^{i q \theta_0}, & \text{for } j > 0. \end{cases} \]
The scattered field may be written in the form

\[
\rho_{\text{scat}}^{[0]}(r) = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B_n^{(0)} e^{ijk_1^i d} H_n^{(1)}(kr_j) e^{in\theta_j} \\
= \sum_{n \in \mathbb{Z}} B_n^{(0)} H_n^{(1)}(kr_0) e^{in\theta_0} \\
+ \sum_{j < 0} \sum_{n \in \mathbb{Z}} B_n^{(0)} e^{ijk_1^i d} H_n^{(1)}(kr_j) e^{in\theta_j} + \sum_{j > 0} \sum_{n \in \mathbb{Z}} B_n^{(0)} e^{ijk_1^i d} H_n^{(1)}(kr_j) e^{in\theta_j}.
\]

For \((r_0 < d - R) \cup \) unit cell

\[
\rho_{\text{scat}}^{[0]}(r) = \sum_{n \in \mathbb{Z}} B_n^{(0)} H_n^{(1)}(kr_0) e^{in\theta_0} \\
+ \sum_{n \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} B_q^{(0)} \sum_{j > 0} H_{n-q}^{(1)}(kd) \left( e^{ijk_1^i d} + (-1)^{n-q} e^{-ijk_1^i d} \right) J_n(kr_0) e^{in\theta_0}
\]

\text{Schlōmilch serie}
Comments on the Schlömilch serie

The serie:

\[ S_n = \sum_{j>0} H_n^{(1)}(kd) \left( e^{ijk_1d} + (-1)^n e^{-ijk_1d} \right) \]

is known to be slowly converging in absence of losses.

A large litterature exists on the numerical evaluation of this serie


...
The incident field reads as

\[ p^i(r_0) = e^{-ik^i r_0 \cos(\theta^0 - \theta^i)} \times \sum_{n \in \mathbb{Z}} (-i)^n J_n(kr_0) \, e^{in(\theta^0 - \theta^i)} = \sum_{n \in \mathbb{Z}} A_{n^i} J_n(kr_0) \, e^{in\theta_0}, \]

so we end with

\[ p^{[0]}(r) = \sum_{n \in \mathbb{Z}} \left( B^{(0)}_n H^{(1)}_n(kr_0) + \left( \sum_{q \in \mathbb{Z}} B^{(0)}_q S_{n-q} + A_{n^i} \right) J_n(kr_0) \right) e^{in\theta_0}. \]

Once again, we have

\[ B^{(1)}_n = SC_n^1 \left( A_{n^i} + \sum_{q \in \mathbb{Z}} B^{(0)}_q S_{n-q} \right), \]

which may be written in matrix form

\[ [\text{Id} - S] \mathbf{B}^0 = \mathbf{A}^0. \]

\[ p^{[0]}(r) = \sum_{n \in \mathbb{Z}} (-i)^n J_n(kr) \, e^{in(\theta - \theta^i)} + \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B^{(j)}_n H^{(1)}_n(kr_j) e^{in\theta_j}. \]

We do not really use the reciprocal space and Bloch waves!
Reflection and Transmission coefficients by an array of rigid cylinders
Scattering by an array of rigid cylinders

Look for $R_q$ and $T_q$

\[
\begin{align*}
p^{[0]+} &= \sum_{q \in \mathbb{Z}} \delta_{q0} e^{ik_1 q x_1 - ik_2 q (x_2 - L)} + R_q e^{ik_1 q x_1 + ik_2 q (x_2 - L)}, \\
p^{[0]-} &= \sum_{q \in \mathbb{Z}} T_q e^{ik_1 q x_1 - ik_2 q x_2},
\end{align*}
\]

where $k_{1q} = k_1^i + \frac{2\pi q}{d}$, $k_{2q} = \sqrt{k^2 - k_{1q}^2}$, with $\Re(k_{2q}) \geq 0$.

Warning: $A_n^{0i} \leftarrow A_n^{0i} e^{ik_2 L}$. 

Note: The diagram shows an array of rigid cylinders with incident and scattered wave directions.
Periodic Green’s function

\[
\left\{
\begin{array}{l}
(\nabla + k^2)G(x, x^s) = -\delta_{x_1^s+jd,x_2^s}, \ j \in \mathbb{Z} \\
G(x, x^s) \sim \text{outgoing waves when } x_2 \to \infty,
\end{array}
\right.
\]

\[
G(x, x^s) = \sum_{j \in \mathbb{Z}} \frac{i}{4\pi} \int_{-\infty}^{\infty} e^{ik_1(x_1-x_1^s-jd)+ik_2|x_2-x_2^s|} \frac{dk_1}{k_2},
\]

with \( k_2 = \sqrt{k^2 - k_1^2} \) and \( \text{Re}(k_2) \geq 0 \), with \( k_{1j} = \frac{2\pi j}{d} \).

Using the Poisson formula

\[
\sum_{j=-\infty}^{\infty} e^{-ik_1jd} = \frac{2\pi}{d} \sum_{j=-\infty}^{\infty} \delta_{k_1q},
\]

we get

\[
G(x, x^s) = \sum_{j \in \mathbb{Z}} \frac{i}{2d} \frac{e^{ik_1q(x_1-x_1^s)+ik_2q|x_2-x_2^s|}}{k_{2q}}.
\]
Scattered field by the grating

From the Green’s theorem we have

\[ p^0(r) = \int_D \left( p^0(r^s) \frac{\partial G(r, r^s)}{\partial r^s} - G(r, r^s) \frac{\partial p^0(r^s)}{\partial r^s} \right) d'D. \]

Using \( p^0(r^s) = \sum_{n \in \mathbb{Z}} \left( B_q^0 H_n^{(1)}(kR) + \sum_{q \in \mathbb{Z}} B_q^0 S_{n-q} J_n(kR) \right) e^{in\theta_0}, \)

orthogonality of \( e^{in\theta} \), and Wronskian identity, we end with

\[ p^{0\pm'}(x) = \sum \sum B_n^0 K_q \cdot e^{-ik_1 x_1 + ik_2 x_2} e^{i k_2 x_2 \mp i k_1 x_1}, \]

with \( K_{qn} = \frac{2(-i)^n}{dk_{2q}} e^{i n \theta_q}, \)

where \( ke^{i \theta_q} = k_1 q + ik_2 q. \)
Identification of $R_q$ and $T_q$

We have

\[
\begin{align*}
\left\{ \begin{array}{l}
p^{[0]+'}(x) = \sum_q \sum_n B_n^{(0)} K_{qn} e^{-ik_1 x_1^0 - i k_2 x_2^0} e^{ik_1 x_1 + ik_2 x_2}, & \text{for } x_2 > x_2^0 + R, \\
p_{\text{refl}}^{[0]+}(x) = \sum_q R_q e^{ik_1 x_1 + i k_2 (x_2 - L)}, & \text{for } x_2 \geq L > x_2^0 + R.
\end{array} \right.
\]

Making use of $\int_0^1 e^{i(k_1 - k_1 m) x_1} dx_1 = 2\pi d \delta_{qm}$, we end with

\[
R_q = \sum_n B_n^{(0)} K_{qn} e^{-ik_1 x_1^0 - i k_2 (x_2^0 - L)}.
\]

On the other hand we have

\[
\begin{align*}
\left\{ \begin{array}{l}
p^{[0]-'}(x) - p^i(x) = \sum_q \sum_n B_n^{(0)} K_{qn} e^{-ik_1 x_1^0 + i k_2 x_2^0} e^{ik_1 x_1 - ik_2 x_2}, & \text{for } x_2 < x_2^0 - R, \\
p^{[0]-}(x) = \sum_q T_q e^{ik_1 x_1 - i k_2 x_2}, & \text{for } x_2 \leq 0 < x_2^0 - R.
\end{array} \right.
\]

Making use of the orthogonality of the Bloch waves, we end with

\[
T_q = \sum_n B_n^{(0)} K_{qn} e^{-ik_1 x_1^0 + i k_2 x_2^0} + \delta_{q0} e^{i k_2 L}.
\]
Field representation in cartesian coordinates in $\Omega^{[0]±}$

Field representation in cylindrical coordinates in $\Omega_{C_0}$

For large radius cylinders, we should run the sum in the direct spatial domain in the red regions...
Far below the possible resonance of the scatterers, both $R_q$ and $T_q$ present a pole when $k_{2q} = 0$. 
In particular, when $k_{21} = 0$, i.e., $k_1^i \pm \frac{2\pi}{d} = k$, all the energy is spread along the grating (at normal incidence $\lambda = d$) and $\alpha = 1$.

This has led several authors to study propagation of this type of guided/surface waves Porter and Evans, J. Fluid Mech., 1999.
Reflection and Transmission by a stack of gratings
Scattering by a stack of array of rigid cylinders

Look for \( p^{[0]}(r) \), \( \forall r \in \Omega^{[0]} \),

\[
\begin{align*}
& (\nabla + k^2)p^{[0]}(r) = 0, \\
& p^{[0]}(r) - p^i(r) \sim \text{outgoing waves},
\end{align*}
\]

wherein \( p^i(r) = e^{ik_1^ix_1 - ik_2^ix_2} \), with \( k_1^i = -k \cos (\theta^i) \) and

\[ k_2^i = \sqrt{k^2 - (k_1^i)^2}, \text{ with } \Re(k_2^i) \geq 0. \]
Scattering by a stack of array of rigid cylinders

Boundary conditions: \( \frac{\partial p[0]}{\partial r} \bigg|_{r=R} = 0. \)

The field is quasi-periodic (*Floquet-Bloch condition*):

\[
p[0](x_1 + nd, x_2) = p[0](x_1, x_2) e^{ik_1 nd}, \quad \forall \vec{x} \in \mathbb{R}^2 \text{ and } \forall n \in \mathbb{Z}.
\]

\( \Rightarrow \) It is sufficient to determine the field in the unit cell \( \mathcal{C} \).
Scattering by a stack of array of rigid cylinders

It exist several ways to solve this problem:

- **Scattering Matrix**: large litterature notably by the group McPhedran and L. Botten
- **Transfer Matrix**
- **Considering the unit cell as a kind of supercell**
Scattering by a stack of array of rigid cylinders

Look for $R_q$ and $T_q$

$$p^{[0]+} = \sum_{q \in \mathbb{Z}} \delta_{q0} e^{ik_1q x_1 - ik_2q(x_2 - L)} + R_q e^{ik_1q x_1 + ik_2q(x_2 - L)}, \ \forall x \in \Omega^{[0]+},$$

$$p^{[0]-} = \sum_{q \in \mathbb{Z}} T_q e^{ik_1q x_1 - ik_2q x_2}, \ \forall x \in \Omega^{[0]-},$$

$$p^{[1]} = \sum_{q \in \mathbb{Z}} \left( f^+_q e^{ik_2q x_2} + f^-_q e^{ik_2q x_2} \right) e^{ik_1q x_1} + \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B_n^{(j)} H_n^{(1)}(kr_j) e^{in\theta_j},$$

where $k_{1q} = k^i_1 + \frac{2\pi q}{d}$, $k_{2q} = \sqrt{k^2 - k_{1q}^2}$, with $\text{Re}(k_{2q}) \geq 0.$
Solution of the scattering problem

Application of the BC on $\Gamma^\pm$

\[ R_q = \sum_{j \in \mathcal{J}} B_n^{(j)} K_{qn} e^{-i k_1 q x_1^0 - i k_2 q (x_2^0 - L)} \]

\[ T_q = \sum_{j \in \mathcal{J}} B_n^{(j)} K_{qn} e^{-i k_1 q x_1^0 + i k_2 q (x_2^0 - L)} + e^{i k_2 q L} \delta_q \]

\[ f_q^+ = 0, \text{ and } f_q^- = e^{i k_2 q L} \delta_q \]

Application of the BC on the cylinders

\[ p_{inc}^{[1]}(r_j) = e^{-i k_2 q (x_2 + L)} \delta_q + \sum_{o > j} \sum_{q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B_n^{(o)} K_{qn} e^{i k_1 q (x_1 - x_1^0) + i k_2 q (x_1 - x_2^0)} + \]

\[ \sum_{o < j} \sum_{q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} B_n^{(o)} K_{qn} e^{i k_1 q (x_1 - x_1^0) - i k_2 q (x_2 - x_2^0)} \]
Solution of the scattering problem

- Application of the BC on $\Gamma^\pm$

\[
R_q = \sum_{j \in \mathcal{J}} B_n^{(j)} K^+_q e^{-ik_1q x_1^0 - ik_2q (x_2^0 - L)}
\]

\[
T_q = \sum_{j \in \mathcal{J}} B_n^{(j)} K^-_q e^{-ik_1q x_1^0 + ik_2q (x_2^0 - L)} + e^{ik_2q L} \delta_q
\]

\[
f_q^+ = 0, \text{ and } f_q^- = e^{ik_2q L} \delta_q
\]

- Application of the BC on the cylinders

- change of coordinate system
  \[
  \tilde{x}_h = x_h - x^j_h, \ h = 1, 2
  \]

- coordinate type: cartesian $\rightarrow$ cylindrical

\[
e^{ik_1q \tilde{x}_1 \pm ik_1q \tilde{x}_2} = \sum_{n \in \mathbb{Z}} J^\pm_{q n} J_n (k_j) e^{in\theta_j},
\]

where $J^\pm_{q n} = (i)^m e^{\pm i\theta_q}$. 
Formally the pressure field for $\min_{o \neq j} (r_j^o - R^{(o)})$ reads as

$$p^{[0]}(r) = \sum_{n \in \mathbb{Z}} \left( B_n^{(0)} H_n^{(1)}(kr_0) + \left( \sum_{l \in \mathbb{Z}} B_j^l S_{n-l}^j + \sum_{o \neq j} \sum_{l \in \mathbb{Z}} B_j^{(o)} S_{n+l}^{(o,j)} + A_0^o \right) J_n(kr_0) \right) e^{i n \theta_0},$$

and we may apply again the relation $B_j^n = SC_j^n A_n$.

In this case $S_{n,l}^{(o,j)} = \sum_{q \in \mathbb{Z}} \frac{2(-i)^{n-l} e^{\pm i(n-l) \theta_q}}{dk_{2q}} e^{i k_{1q}(x_1^j-x_1^o) \pm i k_{2q}(x_2^j-x_2^o)}$, ($+$: $x_2^j > x_2^o$).

More complete expression may be found in Groby et al., J. Acoust. Soc. Am., 11

Final system for the solution of $B_j^n$, $\forall n \in \mathbb{Z}$ and $\forall j \in J$
Example: 7 rows sonic crystal of infinite lateral extend

![Graph showing frequency against distance]

- $d = 10 \, \text{cm}$
- $2r = 7 \, \text{cm}$

**Fabry-Perot interferences**
Band diagram calculation
Various methods

- Plane Wave Expansion (as presented previously by J. Vasseur)
  ⇒ easy to use (eigenvalue problem)
  ⇒ limited to lossless cases and a single type of material per unit cell

- Extended Plane Wave Expansion
  ⇒ easy to use (eigenvalue problem)
  ⇒ single type of material per unit cell

- Method based on the Multiple Scattering Theory
    ⇒ implicit in term of Bloch wave
    ⇒ implicit in term of Bloch wave
Obtention of the eigenvalue problem

We have

\[ p_u = \sum_{q \in \mathbb{Z}} \left( a_{uq} e^{-i k_2 q (x_2 - L)} + a_{uq}^+ e^{i k_2 q (x_2 - L)} \right) e^{i k_1 q x_1} \]

\[ p_d = \sum_{q \in \mathbb{Z}} \left( a_{dq}^+ e^{i k_2 q x_2} + a_{dq}^- e^{-i k_2 q x_2} \right) e^{i k_1 q x_1} \]

The terms \( a_{uq}^\pm \) and \( a_{dq}^\pm \) may be arranged in \( a_u^\pm \) and \( a_d^\pm \).
Obtention of the eigenvalue problem

Because of the orthogonality of the Bloch waves, we write:

\[
\begin{bmatrix}
a_u^+ \\
a_u^-
\end{bmatrix} = e^{i k_{2B} L} \begin{bmatrix}
a_d^+ \\
a_d^-
\end{bmatrix},
\]

where \( k_{2B} \) is the projection of the Bloch wave number \( k_B \) along \( x_2 \), such that \( k_B = \sqrt{(k_1^i)^2 + k_{2B}^2} \).
Obtention of the eigenvalue problem

\[ R^Q_q \text{ and } T^Q_q \text{ might be calculated when the array is solicited by the } Q\text{-th Bloch wave. Therefore, we may construct a matrices } R \text{ and } T, \text{ and we have, again thanks to the orthogonality of the Bloch waves:} \]

\[
\begin{bmatrix}
a^+_u \\
a^-_u
\end{bmatrix} = \begin{bmatrix} T & R \\ 0 & \text{Id} \end{bmatrix} \begin{bmatrix} a^+_d \\
a^-_d
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a^+_d \\
a^-_d
\end{bmatrix} = \begin{bmatrix} \text{Id} & 0 \\ R & T \end{bmatrix} \begin{bmatrix} a^+_u \\
a^-_u
\end{bmatrix}.
\]

So, we end with the following eigenvalue problem

\[
\begin{bmatrix} \text{Id} & 0 \\ R & T \end{bmatrix}^{-1} \begin{bmatrix} T & R \\ 0 & \text{Id} \end{bmatrix} \begin{bmatrix} a^+_d \\
a^-_d
\end{bmatrix} = e^{i k_{2BL}} \begin{bmatrix} a^+_u \\
a^-_u
\end{bmatrix}.
\]
Example of band diagram

\[ \text{Along } \Gamma X, \ k_1^i = 0. \]

⇒ Multiple Scattering might also be used to calculate band diagram
⇒ EquiFrequency Surface: might be complicated along the \(XM\) direction
Multiple Scattering Theory is an efficient tool for
- calculating the response of finite dimension sonic crystals,
- calculating the response of finite depth sonic crystals.

Multiple Scattering Theory might also be used for band diagram calculation

Multiple Scattering Theory might also be used in
- phononic crystals
- metaporous materials
- vibroacoustics
- ...

Multiple Scattering Theory is efficient for cylindrical (ovaidal) shape scatterers
You have to take care of the losses!

\[ R = 1 \text{ mm and } \phi = 0.5, \ \text{Duclos et al., EPJAP, 2009} \]
In the longwavelength limit $\lambda \gg d$, the viscothermal problem reduces to

$$\begin{aligned}
i\omega \begin{bmatrix}
\rho_\parallel & 0 & 0 \\
0 & \rho_\perp & 0 \\
0 & 0 & \rho_\perp
\end{bmatrix} \cdot \mathbf{V} &= \nabla p, \\
i\omega p &= K \nabla \cdot \mathbf{V},
\end{aligned}$$

where (Johnson-Lafarge model)

$$\begin{aligned}
\rho_j &= \frac{\alpha_{\infty j}}{\phi} \left(1 + \frac{1}{i\tilde{\omega}_j} \sqrt{1 + i\tilde{\omega}_j \frac{M_j}{2}}\right), \text{ with } j = \perp, \parallel, \\
K &= \frac{\gamma P_0}{\phi \left(\gamma - (\gamma - 1) \left[1 + \frac{1}{i\tilde{\omega}'} \sqrt{1 + i\tilde{\omega}' \frac{M'}{2}}\right]\right)},
\end{aligned}$$

with $M' = \frac{8k_0'}{\phi \Lambda_j^2}$, $\omega' = \frac{\omega k_0'}{\nu' \phi}$, $M_j = \frac{8\alpha_{\infty j} k_{0j}}{\phi \Lambda_j^2}$, $\omega' = \frac{\omega k_{0j} \alpha_{\infty j}}{\nu' \phi}$, $j = \perp, \parallel$. 
In the longwavelength limit \( \lambda \gg d \), the viscothermal problem reduces to

\[
\begin{cases}
    i\omega \begin{bmatrix}
        \rho_\parallel & 0 & 0 \\
        0 & \rho_\perp & 0 \\
        0 & 0 & \rho_\perp 
    \end{bmatrix} \cdot \mathbf{V} = \nabla p, \\
    i\omega p = K \nabla \cdot \mathbf{V},
\end{cases}
\]

where (Johnson-Lafarge model)

\[
\begin{align*}
    \phi &= \frac{\pi R^2}{d^2}, \\
    \Lambda' &= \frac{R\phi}{1 - \phi}, \\
    \alpha_\infty_\perp &= 2 - \phi, \\
    \Lambda_\perp &= R \frac{\phi(2 - \phi)}{2(1 - \phi)}, \\
    \Lambda_\parallel &= \Lambda', \\
    k_0' &= R^2 \frac{-2\log(1 - \phi) - 2\phi - \phi^2}{8(1 - \phi)}, \\
    k_{0\perp} &= R^2 \frac{-2\log(1 - \phi) - 2\phi - \phi^2}{16(1 - \phi)}, \\
    k_{0\parallel} &= k_0', \\
    \alpha_\infty_\parallel &= 1,
\end{align*}
\]

DENORMS (Designs for Noise Reducing Materials and Structures) aims at designing multifunctional, light and compact noise reducing treatments, which will be used in realistic environments.

DENORMS brings together skills and knowledge of the complementary communities of scientists working on acoustic metamaterials, sonic crystals and conventional acoustic materials across Europe and overseas.

⇒ 3 interacting Working Groups (WG)

- WG1. Modelling of sound interaction with noise reducing materials and structures
- WG2. Experimental techniques
- WG3. Industrial applications

⇒ DENORMS Activities for 2017-2018

- Training School *Experiments on porous materials and acoustic metamaterials*, Le Mans, 4th-6th Dec. 2017
- Workshop *Experimental techniques in porous materials and acoustic metamaterials*, Leuven, 7th-9th Feb 2018

Further information on https://denorms.eu/
Contact denorms@univ-lemans.fr
Welcome to SAPEM 2017

University of Le Mans and the Laboratory of Acoustics are happy to welcome you for the fifth edition of the Symposium on the Acoustics of Poro-Elastic Materials to be held on the 6th, 7th and 8th of December 2017.

The symposium aims:

• presenting the most up-to-date research in modelling, characterisation
• promoting industrial applications of porous materials,
• bringing together researchers and engineers working in adjacent disciplines concerned with porous media,
• discussing future challenges for this area of research.

During this edition, a special focus will be put on metamaterials involving porous media and viscothermal dissipation and on the interaction of porous materials with flow.

Program

The non-parallel sessions of this fifth edition will include:

• Physical models for porous materials
• Experimental methods for porous materials
• Numerical models for porous materials
• Industrial applications
• Metamaterials with porous materials
• Interaction of Porous materials and flows

Special attention will be paid to reserving time for discussion.

A fruitful poster session will be organised alongside with exhibition by our industrial partners.

International Scientific Committee

Keith Attenborough (The Open-University London, UK)  
Yves Aurégin (LAUM - CNRS, France)  
Susan Bui (KTH, Sweden)  
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Abstract submission

There is no requirement to submit a full paper, but presenters should submit an abstract of up to 1,500 words (about two pages), including details of their results and references. The Conference Committee reserves the right to decline submissions that are not in line with the objectives of the Symposium.

Presentations will be collected and compiled in a digital form. Selected papers will be recommended for publication in a special section of a peer-review journal.

Registration

The full registration fee is € 460, € 410 for EAA members and € 310 for students. It covers attendance, instructional materials, proceedings, coffee breaks, lunches and social events. There will be a € 60 discount off the full registration fee for registration made prior to October 20th, 2017.

Registration must be carried out following the link available on the Conference website:

http://sepam2017.mateleys.com

Important dates

Deadline for abstract submissions: Sep. 8th, 2017  
Deadline for early bird registrations: Oct. 20th, 2017  
Deadline for registrations: Nov. 17th, 2017

Contacts

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Thank you for your attention.

Any questions?